ON THE OSCILLATION OF IMPULSIVELY DAMPED HALFLINEAR OSCILLATORS

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Abstract. The authors consider the nonlinear impulsive system

$$(\phi_\beta(x'))' + \phi_\beta(x) = 0 \quad (t \neq t_n), \quad x'(t_n + 0) = b_n x'(t_n)$$

where $n = 1, 2 \ldots$, $\phi_\beta(u) = |u|^\beta \operatorname{sgn} u$ with $\beta > 0$, and $0 \le b_n \le 1$. They investigate the oscillatory behavior of the solutions. In the special case where $b_n = b < 1$ and $t_n = t_0 + n\, d$, they give necessary and sufficient conditions for the oscillation of all solutions.

1. Introduction

Consider the system with impulsive perturbations

$$\begin{aligned} &(\phi_\beta(x'))' + \phi_\beta(x) = 0, \quad t \neq t_n, \\ &x(t_n + 0) = x(t_n), \quad x'(t_n + 0) = b_n x'(t_n), \end{aligned} \tag{1}$$

where $0 \le t_1 < t_2, \ldots, t_n < t_{n+1}$, $t_n \to \infty$ as $n \to \infty$, $0 \le b_n \le 1$ for $n = 1, 2 \ldots$, and $\phi_\beta(u) = |u|^\beta \operatorname{sgn} u$ with $\beta > 0$. System (1) is an impulsive analogue of the nonlinear oscillator equation

$$(\phi_\beta(x'))' + a(t)\phi_\beta(x') + \phi_\beta(x) = 0 \tag{2}$$

with a nonnegative damping coefficient $a(t)$. A detailed description of this analogy can be found in the papers [6, 7, 8] by the present authors. Note that a negative b_n results in a beating effect, which has no continuous analogue (see the discussions of the case $\phi_\beta(u) = u$ in [5, 7]). Systems of the form (1) in the case where $b_n \ge 1$ have been studied, for example, in [10].

1991 *Mathematics Subject Classification.* 34D05, 34D20, 34C15.

Key words and phrases. Oscillation, second order nonlinear systems, impulses.

The research of J. Karsai is supported by Hungarian National Foundation for Scientific Research Grant no. T 029188.

It is known that if the function a is small, but large enough in some sense, then the solutions of (2) oscillate and tend to zero (for example, if $\frac{1}{t} \leq a(t) < 2$ in the linear case $\phi_\beta(u) = u$). This situation is called the small damping case. For large enough $a(t)$, the solutions are monotone decreasing to zero in magnitude, and this is sometimes called the large damping case (for example, if $2 \leq a(t) < t$ and $\phi_\beta(u) = u$). If the system is overdamped, that is, $a(t)$ grows too fast to infinity as t tends to infinity (for example, $t^{1+\varepsilon} \leq a(t)$ in the case $\phi_\beta(u) = u$), then the solutions decrease in magnitude but may not tend to zero (see [2, 11, 12] and the references therein).

The problem of attractivity for system (1) and its special cases was investigated in [5, 6, 7, 8, 9]. In [7] and [9], conditions are given to ensure that every solution of system (1) is nonoscillatory, and these conditions turn out to be necessary and sufficient in case $\phi_\beta(u) = u$, $b_n = b$, and $t_n = t_0 + n\,d$.

In this paper, we improve the method applied in [6, 9] and formulate new conditions for the oscillation and nonoscillation of the solutions, and these will result in a necessary and sufficient condition for the system with constant impulses at equally spaced distances.

2. Preliminaries

We know that every solution of (1) can be continued to ∞, the solutions are piecewise differentiable, and $\phi_\beta(x'(t))$ is piecewise continuous and continuous from the left hand side at every $t > 0$. The following result classifies the solutions of the system (1) as being either monotonic or oscillatory on some interval $[T, \infty)$.

Lemma 1. ([9]) *Suppose that $0 \leq b_n \leq 1,\ n = 1, 2, \ldots$. Let $x(t)$ be a solution of (1) that is not identically zero on any interval $[T, \infty)$, and let s_1 and s_2 be consecutive zeros of $x'(t)$. Then there exists $\tilde{t} \in (s_1, s_2)$ such that $x(\tilde{t}) = 0$. Hence, solutions of (1) are either oscillatory or monotone nonincreasing in magnitude.*

Define the energy function

$$V(x, y) = y\,\phi_\beta(y) - \int_0^y \phi_\beta(s)\,ds + \int_0^x \phi_\beta(s)\,ds =: \Phi_\beta(y) + F_\beta(x), \quad (3)$$

where in explicit form $\Phi_\beta(y) = \frac{\beta}{\beta+1}|y|^{\beta+1}$ and $F_\beta(x) = \frac{1}{\beta+1}|x|^{\beta+1}$. Note that the functions F_β and Φ_β are both even and positive definite.

The function $V(t) = V(x(t), x'(t))$ is constant along the solutions of the equation without impulses

$$(\phi_\beta(x'))' + \phi_\beta(x) = 0. \tag{4}$$

Since the solutions of (1) are identical with those of (4) between the instants of impulses, some basic knowledge of their behavior will be useful here. It is easy to see, for example, that every nonzero solution of (4) is periodic. In addition, since we have assumed that ϕ_β is odd, the length of the time intervals on which $x(t)x'(t) \geq 0$ or $x(t)x'(t) \leq 0$ are equal. The distance between two consecutive zeros of $x'(t)$, i.e., the *half-period*, will be equal for every solution due to the fact that the equation (4) is homogeneous and autonomous (see [3]). We denote the half-period by Δ_β. A formula for Δ_β can be obtained as a special case of the following lemma.

Lemma 2. *Let $x(t)$ be a solution of (4) with $V(t) \equiv r > 0$, and let $\tau_1 < \tau_2$ be such that $F_\beta(\tau_1) = \delta r$, $F_\beta(\tau_2) = \gamma r$, $0 < \tau_2 - \tau_1 < \Delta_\beta/2$, and $0 \leq \delta < \gamma \leq 1$. Then, the time, $\tau_2 - \tau_1$, elapsed by changing F_β from δr to γr can be expressed by the following integral:*

$$\tau_2 - \tau_1 = \int_\delta^\gamma \frac{dv}{\phi_\beta(F_\beta^{-1}(v))\Phi_\beta^{-1}(1-v)} = \frac{\beta^{\frac{1}{1+\beta}}}{1+\beta}\int_\delta^\gamma \frac{dv}{(1-v)^{\frac{1}{1+\beta}}\, v^{\frac{\beta}{1+\beta}}}. \tag{5}$$

Although the above lemma is proved in [10], the proof itself is short and contains basic arguments concerning the solutions, so we provide it below.

Proof. Let $x(t)$ be a solution of (4) with $V(t) \equiv r$, $F_\beta(x(\tau_1)) = \delta r$, and $F_\beta(x(\tau_2)) = \gamma r$. From (3), we have $x'(t) = \Phi_\beta^{-1}(r - F_\beta(x(t)))$. Dividing by the right-hand side and integrating, we obtain

$$\tau_2 - \tau_1 = \int_{\tau_1}^{\tau_2} \frac{x'(t)\,dt}{\Phi_\beta^{-1}(r - F_\beta(x(t)))}. \tag{6}$$

Making the substitution $x = x(t)$, $\tau_2 - \tau_1$ can be expressed in the form

$$\tau_2 - \tau_1 = \int_{F_\beta^{-1}(\delta r)}^{F_\beta^{-1}(\gamma r)} \frac{dx}{\Phi_\beta^{-1}(r - F_\beta(x))} = \int_\delta^\gamma \frac{dv}{\phi_\beta(F_\beta^{-1}(v))\Phi_\beta^{-1}(1-v)} \tag{7}$$

where $u = F_\beta(x)$, $v = u/r$, and F_β^{-1} and Φ_β^{-1} are the inverses of F_β and Φ_β on $[0, \infty)$, respectively. □

To simplify the formulation of our results, we will use the following notation:

$$H_\beta(v) := \frac{1}{\phi_\beta(F_\beta^{-1}(v))\Phi_\beta^{-1}(1-v)}. \tag{8}$$

Taking $\delta = 0$ and $\gamma = 1$ in (5), we obtain the following expression for Δ_β:

$$\Delta_\beta = \frac{2\,\beta^{\frac{1}{1+\beta}}\,\Gamma(\frac{1}{1+\beta})\,\Gamma(\frac{\beta}{1+\beta})}{1+\beta} = \frac{2\pi\beta^{\frac{1}{1+\beta}}}{(1+\beta)\sin\frac{\pi}{1+\beta}}, \tag{9}$$

and in particular for the linear case $\beta = 1$, we have $\Delta_1 = \pi$.

Now, let $x(t)$ be a solution of (1). The jump in the energy along $x(t)$ at t_n is given by

$$\begin{aligned} V(t_{n+1}) - V(t_n) &= V(t_n+0) - V(t_n) \\ &= \Phi_\beta(x'(t_n+0)) + F_\beta(x(t_n+0)) - \Phi_\beta(x'(t_n)) - F_\beta(x(t_n)) \\ &= b_n^{\beta+1}\Phi_\beta(x'(t_n)) - \Phi_\beta(x'(t_n)) = \Phi_\beta(x'(t_n))(b_n^{\beta+1}-1). \end{aligned} \tag{10}$$

Denoting $r_n = V(t_{n-1}+0) = V(t_n - 0)$ and $F_\beta(x(t_n)) = \sigma_n r_n$, and calculating $F_\beta(x(t_n))$ in terms of $r_{n+1} = V(t_n+0)$, we obtain

$$\begin{aligned} r_{n+1} &= F_\beta(x(t_n)) + b_n^{\beta+1}\Phi_\beta(x'(t_n-0)) \\ &= F_\beta(x(t_n)) + b_n^{\beta+1}(r_n - F_\beta(x(t_n))) \\ &= b_n^{\beta+1} r_n + F_\beta(x(t_n))(1-b_n^{\beta+1}) \\ &= b_n^{\beta+1} r_n + (1-b_n^{\beta+1}) r_n\sigma_n \\ &= r_n\left[b_n^{\beta+1} + (1-b_n^{\beta+1})\sigma_n\right]. \end{aligned}$$

Hence,

$$F_\beta(x(t_n)) = \frac{\sigma_n}{b_n^{\beta+1} + (1-b_n^{\beta+1})\sigma_n}\, r_{n+1}.$$

In order to simplify the notation, we let

$$\Theta(u,b) := \frac{u}{b^{\beta+1}(1-u)+u}. \tag{11}$$

The function Θ measures the jump in the quantity $F_\beta(x(t))/V(t)$. It is clear that $\Theta(0,b) = 0$, $0 < u < \Theta(u,b)$ for $0 < u < 1$ and $b \le 1$, and that Θ is monotone increasing with respect to u and decreasing with respect to b.

3. Oscillation and Nonoscillation Results

Our main nonoscillation theorem is as follows.

Theorem 3. *Assume there exist a constant $N > 0$ and a sequence $\{\gamma_n\}$ with $0 < \gamma_{n+1} \le \Theta(\gamma_n, b_n) < 1$ such that*

$$t_{n+1} - t_n \le \int_{\gamma_{n+1}}^{\Theta(\gamma_n,b_n)} H_\beta(v)\,dv \tag{12}$$

holds for every $n > N$. Then every solution of (1) is nonoscillatory and

$$\frac{F(x(t_n))}{V(t_n - 0)} \ge \gamma_n.$$

Proof. Let $x(t)$ be a nontrivial solution of (1). Clearly, the trajectory of $x(t)$ cannot remain in either the first or the third quadrant, i.e., $x(t)x'(t) \ge 0$ cannot hold on any half-ray $[T, \infty)$. We will show that $x(t)x'(t) < 0$ for $t \ge T$ for some $T > 0$.

Now $V(t) = F_\beta(x(t)) + \Phi_\beta(x'(t))$ is constant on each interval (t_{n-1}, t_n), and we denote this value by r_n. Define σ_n by $F_\beta(x(t_n)) = \sigma_n r_n$. If we can show that

$$0 < \gamma_n r_n \le \sigma_n r_n \tag{13}$$

holds for sufficiently large n, this would imply the nonoscillation of $x(t)$.

Letting s_0 be a zero of $x'(t)$ with $t_{n-1} \le s_0 < t_n$, we can assume that $x(s_0) > 0$; otherwise, we can consider $-x(t)$ and use the symmetry of the equation. Now (5) and (12) imply

$$\begin{aligned} t_n - s_0 &= \int_{\sigma_n}^{1} H_\beta(v)\,dv \le t_n - t_{n-1} \\ &\le \int_{\gamma_n}^{\Theta(\gamma_{n-1},b_{n-1})} H_\beta(v)\,dv < \int_{\gamma_n}^{1} H_\beta(v)\,dv, \end{aligned}$$

which, by the monotonicity of the integral on the right-hand-side of the above inequality, implies (13) holds. From the definition of the function Θ, we have

$$F_\beta(x(t_n)) = \Theta(\sigma_n, b_n) r_{n+1}.$$

If we define the function $h_n(u)$ implicitly by

$$t_{n+1} - t_n = \int_{h_n(u)}^{u} H_\beta(v)\,dv,$$

a differentiation shows that $h_n(u)$ is monotone decreasing. Clearly,

$$F_\beta(x(t_{n+1})) = h_n(\Theta(\sigma_n, b_n))r_{n+1}.$$

From the monotonicity of Θ, we obtain

$$\begin{aligned} t_{n+1} - t_n &= \int_{\sigma_{n+1}}^{\Theta(\sigma_n,b_n)} H_\beta(v)\,dv \\ &= \int_{h_n(\Theta(\gamma_n,b_n))}^{\Theta(\gamma_n,b_n)} H_\beta(v)\,dv \le \int_{\gamma_{n+1}}^{\Theta(\gamma_n,b_n)} H_\beta(v)\,dv. \end{aligned}$$

Finally, the monotonicity of h_n implies $\sigma_{n+1} \ge \gamma_{n+1}$, and this proves that $x(t)$ is nonoscillatory. □

Theorem 3 can be applied to particular problems by finding appropriate sequences $\{\gamma_n\}$. Let $0 < \gamma < \frac{1}{2}$ be given such that $\Theta(\gamma, b_n) \ge 1 - \gamma$ and

$$t_{n+1} - t_n \le \int_\gamma^{1-\gamma} H_\beta(v)\,dv, \quad n = 0, 1, 2, ...;$$

then (12) holds ([9]).

Note that any choice $0 < \gamma_n = \gamma < 1$ results in a nonoscillation criterion, but to formulate sharp conditions, we need some further investigation.

Next, we find a monotone nonincreasing sequence $\{\gamma_n\}$ for which (12) holds. In this case, the integral in (12) is estimated from below by

$$\int_{\gamma_n}^{\Theta(\gamma_n,b_n)} H_\beta(v)\,dv.$$

Applying the usual methods, we obtain that this integral takes its maximum at

$$\bar\gamma_n = \frac{b_n^\beta(b_n - 1)}{b_n^{\beta+1} - 1}. \tag{14}$$

Note that $\bar\gamma_n$ is monotone increasing with respect to b_n. If the sequence $\{b_n\}$ is nonincreasing, then $\bar\gamma_n$ is also nonincreasing. Hence, in this case $\gamma_n := \bar\gamma_n$ gives a better estimate in (12). In general, a nonincreasing $\{\gamma_n\}$ can be defined by $\gamma_n := \min_{i=1,\dots,n} \bar\gamma_i$. In particular, if $b_n \le b < 1$, then we can apply

$$\gamma_n := \bar\gamma = \frac{b^\beta(b - 1)}{b^{\beta+1} - 1}. \tag{15}$$

On the other hand, numerical simulations show that for the case $\lim_{n\to\infty} b_n = 1$, the sharpest criterion can be obtained by choosing

$$\gamma_n = \lim_{b\to 1} \frac{b^\beta(b-1)}{b^{\beta+1}-1} = \frac{1}{1+\beta}.$$

Summarizing the above arguments, we can formulate the following corollary.

Corollary 4. *Let the sequence γ_n be defined as follows:*
If the sequence $\{b_n\}$ is nonincreasing, let $\gamma_n := \bar{\gamma}_n$.
If $b_n \le b < 1$, let $\gamma_n := \bar{\gamma}$.
If $\lim_{n\to\infty} b_n = 1$, let $\gamma_n := \frac{1}{1+\beta}$.
Otherwise, let $\gamma_n := \gamma \in (0,1)$.
If (12) holds, then every solution of (1) is nonoscillatory.

Next, we prove our main oscillation result.

Theorem 5. *Assume that there exist a constant $N > 0$ and a sequence $\{\lambda_n\}$ with $0 < \lambda_n \le 1$ and $\sum_{n=1}^\infty \lambda_n = \infty$ such that for every $\gamma \in [\lambda_n, 1]$,*

$$t_{n+1} - t_n \ge \int_{\gamma-\lambda_n}^{\Theta(\gamma,b_n)} H_\beta(v)\, dv \tag{16}$$

holds for every $n > N$. Then every solution of (1) is oscillatory.

Proof. Let $x(t)$ be a nontrivial solution of (1). It will suffice to show that $x(t)x'(t) < 0$ cannot hold on any interval $[T,\infty)$, so suppose $x(t) > 0$ and $x'(t) < 0$ for $t \in [t_N,\infty)$. Again, we let σ_n be defined by $F_\beta(x(t_n)) = \sigma_n r_n$, where $r_n = V(t_n - 0)$. It follows that $\sigma_n > \lambda_n$ since, in the opposite case, (16) yields

$$t_{n+1} - t_n \ge \int_0^{\Theta(\sigma_n,b_n)} H_\beta(v)\, dv,$$

and hence $x(t_{n+1}) \le 0$, which contradicts to the positivity of $x(t)$.

Now assuming $\sigma_n > \lambda_n,\ n = N, ...$, it follows again from (16) that $0 < \sigma_{n+1} \le \sigma_n - \lambda_n$. Hence,

$$\sigma_{n+1} \le \sigma_N - \sum_{i=N}^{n} \lambda_n,$$

and since the right-hand-side tends to negative infinity as n tends to infinity, this contradicts the positivity of $F_\beta(x(t))$. □

If the sequence $\{b_n\}$ is bounded away from zero, the following corollary holds.

Corollary 6. *Assume that* $0 < b \le b_n \le 1$ *and there exist a constant* $N > 0$ *such that the sequence* $\{\mu_n\}$ *defined by*

$$\mu_n := (t_{n+1} - t_n) - \int_{\bar{\gamma}}^{\Theta(\bar{\gamma},b)} H_\beta(v)\, dv \ge 0 \tag{17}$$

satisfies $\sum_{n=N}^{\infty} \mu_n = \infty$*, where* $\bar{\gamma}$ *is defined by (15). Then every solution of (1) is oscillatory. In particular, if* $t_{n+1} - t_n \ge d > 0$ *and*

$$d > \int_{\bar{\gamma}}^{\Theta(\bar{\gamma},b)} H_\beta(v)\, dv \tag{18}$$

holds for every $n > N$*, then every solution of (1) is oscillatory.*

Proof. We will find a sequence λ_n that satisfies (16). The integral on the right-hand-side of (16) can be estimated from above by replacing b_n with $b \le b_n$. For a given λ_n, let $\gamma_n' \in [0, 1]$ be the place where the value

$$\max_{\gamma \in [0,1-\lambda_n]} \left(\int_{\gamma}^{\Theta(\gamma+\lambda_n,b)} H_\beta(v)\, dv \right)$$

is attained. Let us define λ_n implicitly by the relation

$$\int_{\gamma_n'-\lambda_n}^{\Theta(\gamma_n',b)} H_\beta(v)\, dv = \min(\mu_n, \frac{1}{n}) + \int_{\bar{\gamma}}^{\Theta(\bar{\gamma},b)} H_\beta(v)\, dv. \tag{19}$$

By continuity arguments, we obtain that $\lim_{n\to\infty} \gamma_n' = \bar{\gamma}$ and $\lim_{n\to\infty} \lambda_n = 0$. Hence, for sufficiently large n we have that $\gamma_n' \in [\varepsilon, 1 - \varepsilon]$ for some $\varepsilon \in (0, 1/2)$. Since the integral on the left-hand-side of (19) is Lipschitzian with respect to λ_n uniformly in $\gamma_n' \in [\varepsilon, 1-\varepsilon]$, $\sum_N^\infty \mu_n = \infty$ implies $\sum_N^\infty \lambda_n = \infty$, and this proves the statement. □

Without assuming $b_n \ge b > 0$, we can state the following corollary.

Corollary 7. *Assume that there exist a constant* $N > 0$ *such that the sequence* $\{\mu_n\}$ *defined by*

$$\mu_n := (t_{n+1} - t_n) - \int_{\bar{\gamma}_n}^{\Theta(\bar{\gamma}_n,b_n)} H_\beta(v)\, dv \ge 0 \tag{20}$$

satisfies $\sum_{n=N}^{\infty} \mu_n^{1+\beta} = \infty$ *for* $\beta \ge 1$ *and* $\sum_{n=N}^{\infty} (\mu_n)^{(1+\beta)/\beta} = \infty$ *for* $0 < \beta \le 1$*, respectively, where* $\bar{\gamma}_n$ *is defined by (14). Then every solution of (1) is oscillatory.*

Proof. Similar to the proof of Corollary 6, we will find a sequence λ_n that satisfies (16). The integral on the right-hand-side of (16) can be estimated from above by

$$\int_{\bar{\gamma}_n}^{\Theta(\bar{\gamma}_n,b_n)} H_\beta(v)\,dv + \sup_{\gamma\in[\lambda_n,1]} \int\limits_{\gamma-\lambda_n}^{\gamma} H_\beta(v)\,dv.$$

It can be shown that the second term is not greater than $\int_0^{\lambda_n} H_\beta(v)\,dv$ for $\beta \geq 1$ and is not greater than $\int_{1-\lambda_n}^{1} H_\beta(v)\,dv$ for $0 < \beta \leq 1$. Consider the case $\beta \geq 1$. Let us define λ_n implicitly by

$$\int\limits_0^{\lambda_n} H_\beta(v)\,dv = \mu_n. \tag{21}$$

Since

$$\int\limits_0^{\lambda} H_\beta(v)\,dv = \frac{\beta^{\frac{1}{\beta+1}}\ B_\lambda\big(\frac{1}{\beta+1},\frac{\beta}{\beta+1}\big)}{\beta+1} = O(\lambda^{\frac{1}{1+\beta}}),\ \ \lambda\to 0,$$

where $B_z(a,b) = \int_0^z t^{a-1}\,(1-t)^{b-1}\,dt$ is the incomplete Beta function, the conditions $\sum_N^\infty \mu_n^{1+\beta} = \infty$ and $\sum_N^\infty \lambda_n = \infty$ are equivalent, and this part is proved.

For the case $0 \leq \beta \leq 1$, we have only to observe that

$$\int\limits_{1-\lambda}^{1} H_\beta(v)\,dv = \int\limits_0^{\lambda} H_{1/\beta}(v)\,dv = O(\lambda^{\frac{\beta}{1+\beta}}),\ \ \lambda\to 0.$$

□

In the special case $b_n = b$ and $t_{n+1} - t_n = d$ for $n = 1, 2, ...$, combining Corollaries 4 and 6 gives a necessary and sufficient condition for nonoscillation.

Theorem 8. *Assume that $b_n = b$ and $t_{n+1} - t_n = d$ for $n = 1, 2, \ldots$. Every solution of (1) is nonoscillatory if and only if*

$$d \leq \int_{\bar{\gamma}}^{\Theta(\bar{\gamma},b)} H_\beta(v)\,dv. \tag{22}$$

4. The linear case

Now, let us apply the results in the previous section to the linear case ($\beta = 1$)

$$\begin{aligned} &x'' + x = 0, \quad t \neq t_n, \\ &x(t_n + 0) = x(t_n), \quad x'(t_n + 0) = b_n x'(t_n), \end{aligned} \tag{23}$$

We have

$$\int_u^{\Theta(u,b)} H_\beta(v)\, dv = -\arcsin(\sqrt{u}) + \arcsin\left(\sqrt{\frac{u}{b^2(1-u)+u}}\right).$$

Hence,

$$\sin\left(\int_u^{\Theta(u,b)} H_\beta(v)\, dv\right) = (1-b)\,\sqrt{\frac{(1-u)\;u}{b^2\;(1-u)+u}}. \tag{24}$$

Applying Corollary 4 we obtain the following statement.

Corollary 9. *Every solution of system (23) is nonoscillatory if there exists a number* $0 < \gamma < 1$ *such that*

$$\sin(t_{n+1} - t_n) \leq (1 - b_n)\,\sqrt{\frac{(1-\gamma)\;\gamma}{b_n^2\;(1-\gamma)+\gamma}}. \tag{25}$$

In particular, every solution of system (23) is nonoscillatory in the following cases:
a) $b_n \leq b < 1$ *and*

$$\sin(t_{n+1} - t_n) \leq \frac{1-b}{1+b}.$$

b) $b_{n+1} \leq b_n < 1$ *and*

$$\sin(t_{n+1} - t_n) \leq \frac{1-b_n}{1+b_n}.$$

c) $\lim_{n\to\infty} b_n = 1$ *and*

$$\sin(t_{n+1} - t_n) \leq (1 - b_n)\,\sqrt{\frac{1}{2+2\,b_n^2}}.$$

For oscillation, the following corollaries are consequences of Corollaries 6 and 7 respectively.

Corollary 10. *Assume that* $0 < b \le b_n \le 1$, *and let*

$$\mu_n := t_{n+1} - t_n - \arcsin \frac{1-b}{1+b} \ge 0.$$

If $\sum_{n=1}^{\infty} \mu_n = \infty$, *then every solution of (23) is oscillatory. In particular, if* $t_{n+1} - t_n > d > 0$ *and*

$$\sin d > \frac{1-b}{1+b},$$

then every solution of (23) is oscillatory.

Corollary 11. *Let*

$$\mu_n := t_{n+1} - t_n - \arcsin \frac{1-b_n}{1+b_n} \ge 0.$$

If $\sum_{n=1}^{\infty} \mu_n^2 = \infty$, *then every solution of (23) is oscillatory.*

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